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Four identities related to third order mock theta functions in Ramanujan's lost notebook

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Abstract

We prove, for the first time, a series of four related identities from Ramanujan's lost notebook. The identities are connected with third order mock theta functions.

1. Introduction

In his last letter to Hardy, Ramanujan introduced mock theta functions [9, pp. 127–131]. Included in this letter were the four third order mock theta functions:

$$\widetilde{f}(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}, \tag{1.1}$$

$$\widetilde{\phi}(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q^2)(1+q^4)\cdots(1+q^{2n})}, \tag{1.2}$$

$$\widetilde{\psi}(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^3)\cdots(1-q^{2n-1})}, \tag{1.3}$$

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$$\widetilde{\chi}(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(1-q+q^2)(1-q^2+q^4)\cdots(1-q^n+q^{2n})}. \tag{1.4}$$

They satisfy the equations

$$2\widetilde{\phi}(-q)-\widetilde{f}(q)=\widetilde{f}(q)+4\widetilde{\psi}(-q)=(-q;q)_{\infty}^{-1}\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}, \tag{1.5}$$

$$4\widetilde{\chi}(q)-\widetilde{f}(q)=3(q;q)_{\infty}^{-1}\left\{\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2}\right\}^2, \tag{1.6}$$

where we used the standard notation

$$(a;q)_n=(a)_n:=(1-a)(1-aq)\dots(1-aq^{n-1}),$$

$$(a;q)_{\infty}=\prod_{n=0}^{\infty}(1-aq^n), \quad |q|<1.$$

Watson [10] proved (1.5) and (1.6). Andrews [1] also gave certain generalizations of (1.5) and (1.6). Third order mock theta functions are related to the rank of a partition defined by Dyson [5] as the largest part minus the number of parts. Let us define $N(m,n)$ as the number of partitions of n with rank m. The generating function for $N(m,n)$ is given by

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}N(m,n)q^n t^m=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(tq)_n(t^{-1}q)_n}, |q|<1,\ |q|<|t|<|1/q|. \tag{1.7}$$

The third order mock theta functions defined by (1.1) through (1.4) can be expressed in terms of this generating function. Third order mock theta functions and their applications to the rank are detailed by Fine [7]. A comprehensive literature survey on mock theta functions is given by Andrews [2].

We prove, for the first time, a series of four related identities from Ramanujan's lost notebook. These identities are defined and their connections to (1.5) and (1.6) are given in Section 3. Proofs of these identities are provided in Sections 4–7. In addition, we will show in Section 8 that one of the identities can be used to prove the following identity:

$$\frac{(q)_{\infty}^2}{(t)_{\infty}(t^{-1}q)_{\infty}}=\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{n(n+1)/2}}{1-tq^n}. \tag{1.8}$$

Identity (1.8) was proved by Evans [6, Eq. (3.1)] following a different approach. Equality (1.8) is also given in a different form by Ramanujan on p. 59 of the lost notebook [9]. Partition theory implications of the product

$$\frac{(q)_{\infty}}{(tq)_{\infty}(t^{-1}q)_{\infty}}$$

are discussed in [8].

2. Definitions and preliminary results

We first recall Ramanujan's definitions for a general theta function and some of its important special cases. Set

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \tag{2.1}$$

Basic properties satisfied by $f(a,b)$ include [4, p. 34, Entry 18]

$$f(a,b) = f(b,a), \tag{2.2}$$

$$f(1,a) = 2f(a,a^3), \tag{2.3}$$

$$f(-1,a) = 0 \tag{2.4}$$

and if n is an integer,

$$f(a,b) = a^{n(n+1)/2} b^{n(n-1)/2} f(a(ab)^n, b(ab)^{-n}). \tag{2.5}$$

If $n = 1$, (2.5) becomes

$$f(a,b) = af(a^{-1}, a^2 b). \tag{2.6}$$

Two other formulas satisfied by $f(a,b)$ are [4, p. 46, Entry 30]

$$f(a,b) + f(-a,-b) = 2f(a^3 b, ab^3), \tag{2.7}$$

$$f(a,b) - f(-a,-b) = 2af(ba^{-1}, ab^{-1}a^4 b^4). \tag{2.8}$$

The function $f(a,b)$ satisfies the well-known Jacobi triple product identity [4, p. 35, Entry 19]

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty. \tag{2.9}$$

Some important special cases of (2.1) and (2.9) are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty, \tag{2.10}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = (-q;q)_\infty^2 (q;q)_\infty, \tag{2.11}$$

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{2.12}$$

By using (2.10) and (2.11), and elementary product manipulations, we find that

$$\psi(-q) = \frac{(q;q)_\infty}{(q^2;q^4)_\infty} = \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}, \tag{2.13}$$

$$\varphi(-q) = \frac{(q;q)_\infty}{(-q;q)_\infty}. \tag{2.14}$$

Also, after Ramanujan define

$$\chi(q) := (-q;q^2)_\infty. \tag{2.15}$$

Other basic properties of the functions φ, ψ, f and χ are [4, p. 39, Entry 24]

$$\frac{f(q)}{f(-q)} = \frac{\psi(q)}{\psi(-q)} = \frac{\chi(q)}{\chi(-q)} = \sqrt{\frac{\varphi(q)}{\varphi(-q)}}, \tag{2.16}$$

$$\chi(q) = \frac{f(q)}{f(-q^2)} = \sqrt[3]{\frac{\varphi(q)}{\psi(-q)}} = \frac{\varphi(q)}{f(q)} = \frac{f(-q^2)}{\psi(-q)}, \tag{2.17}$$

$$f^3(-q^2) = \varphi(-q)\psi^2(q). \tag{2.18}$$

Combining (2.17) and (2.18), we obtain

$$f^3(-q) = \psi(q)\varphi^2(-q). \tag{2.19}$$

We will frequently use Euler's identity

$$(-q;q)_\infty = (q;q^2)_\infty^{-1}. \tag{2.20}$$

For any real number a, let

$$f_a(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+aq+q^2)\cdots(1+aq^n+q^{2n})}, \tag{2.21}$$

where $|q|<1$. For $|q|<1$, $|q|<|t|<|q|^{-1}$, let

$$G(t,q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(tq)_n(t^{-1}q)_n}. \tag{2.22}$$

We need Euler's famous generating function for partitions,

$$G(1,q) = (q;q)_\infty^{-1}. \tag{2.23}$$

For a proof of (2.23) see [7, p. 13, eq. (12.311)]. We need variations of two representations for $G(t,q)$ due to Fine [7].

Lemma 2.1. *For* $|t|<1$,

$$G(t,q) = (1-t)\sum_{n=0}^{\infty} \frac{t^n}{(t^{-1}q)_n} \tag{2.24}$$

$$= 1 - t^{-1} + t^{-1}\sum_{n=0}^{\infty} \frac{(tq)^n}{(t^{-1}q)_n}. \tag{2.25}$$

Proof. Following Fine [7, pp. 1, Eq. (1.1)], we define

$$F(a,b;t) := \sum_{n=0}^{\infty} \frac{(aq)_n}{(bq)_n} t^n.$$

In this notation, Lemma 2.1 can be written as

$$G(t,q) = (1-t)F(0,t^{-1};t) \tag{2.26}$$

$$= 1 - t^{-1} + t^{-1}F(0,t^{-1};tq). \tag{2.27}$$

Eq. (2.26) is Eq. (12.3) on p. 13 of [7] with b replaced by t^{-1}, and (2.27) readily follows from Eq. (2.4) on p. 2 of [7].

Observe that (2.25) is valid in the region $|q|<|t|<|q|^{-1}$. Also as noted by Fine [7, p. 51, Eq. (25.6)], $G(t,q)$ satisfies a third order q-difference equation. We sketch a proof here since it is stated without a proof in [7].

Lemma 2.2. *For* $|q|<1$ *and* $|q|<|t|<|1/q|$, $G(t,q)$ *satisfies the q-difference equation*

$$\frac{1}{1-tq}G(tq,q) + \frac{qt^3}{1-t}G(t,q) = 1 - qt^2. \tag{2.28}$$

Proof. Let

$$M(t,q) := \sum_{n=0}^{\infty} \frac{(tq)^n}{(t^{-1}q)_n}, \tag{2.29}$$

so that by (2.25),

$$G(t,q) = 1 - t^{-1} + t^{-1}M(t,q). \tag{2.30}$$

Using definition (2.29) and algebraic manipulation, we obtain

$$\begin{aligned}
M(t,q) &= \sum_{n=0}^{\infty} \frac{(tq)^n}{(t^{-1}q)_n} = \sum_{n=0}^{\infty} \frac{(tq)^n(1-t^{-1}q^{n+1})}{(t^{-1}q)_{n+1}} \\
&= \sum_{n=0}^{\infty} \frac{(tq)^n}{(t^{-1}q)_{n+1}} - t^{-1}q\sum_{n=0}^{\infty} \frac{(tq^2)^n}{(t^{-1}q)_{n+1}} \\
&= \sum_{n=0}^{\infty} \frac{(tq)^n}{(t^{-1}q)_{n+1}} - t^{-1}q(1-t^{-1})\sum_{n=0}^{\infty} \frac{(tq^2)^n}{(t^{-1})_{n+2}} \\
&= \frac{1}{tq}\sum_{n=0}^{\infty} \frac{(tq)^{n+1}}{(t^{-1}q)_{n+1}} - \frac{1}{(tq^2)^2}t^{-1}q(1-t^{-1})\sum_{n=0}^{\infty} \frac{(tq^2)^{n+2}}{(t^{-1})_{n+2}} \\
&= \frac{1}{tq}(M(t,q)-1) - \frac{1-t^{-1}}{t^3q^3}\left(M(tq,q)-1-\frac{tq^2}{1-t^{-1}}\right). \qquad (2.31)
\end{aligned}$$

Now, Lemma 2.2 follows from (2.31) together with (2.30) after rearrangement.

For convenience, define

$$V(t,q) := \frac{1}{1-t}G(t,q). \qquad (2.32)$$

Lemma 2.2 then takes the following form:

$$V(tq,q) + qt^3V(t,q) = 1 - qt^2. \qquad (2.33)$$

Observe that

$$V(t^{-1},q) = -tV(t,q). \qquad (2.34)$$

The basic property (2.34) will be used many times in the sequel without comment. The partial fraction decomposition of $V(t,q)$ is given by [8, Eq. (7.10)]

$$V(t,q) = 1 + \frac{t}{(q)_\infty}\sum_{n=-\infty}^{\infty} (-1)^n\frac{q^{3n(n+1)/2}}{1-tq^n}. \qquad (2.35)$$

We will need the following lemma due to Atkin and Swinnerton-Dyer [3].

Lemma 2.3. *Let q, $0<q<1$, be fixed. Suppose that $\vartheta(z)$ is an analytic function of z, except for possibly a finite number of poles, in every region, $0<z_1\leqslant|z|\leqslant z_2$.*

If

$$\vartheta(zq) = Az^k\vartheta(z)$$

for some integer k (positive, zero, or negative) and some constant A, then either $\vartheta(z)$ has k more poles than zeros in the region $|q|<|z|\leqslant 1$, or $\vartheta(z)$ vanishes identically.

3. Four identities of Ramanujan

We now offer the four identities from Ramanujan's lost notebook that we plan to prove.

Entry 3.1 (Ramanujan [9, p. 2, no. 3]). *Suppose that a and b are real with* $a^2+b^2=4$. *Then, if* $f_a(q)$ *is defined by* (2.21),

$$\frac{b-a+2}{4}f_a(-q)+\frac{b+a+2}{4}f_{-a}(-q)-\frac{b}{2}f_b(q)$$
$$=\frac{(q^4;q^4)_\infty}{(-q;q^2)_\infty}\prod_{n=1}^{\infty}\frac{1-bq^n+q^{2n}}{1+(a^2b^2-2)q^{4n}+q^{8n}}. \tag{3.1}$$

If we take $a=0$ and $b=2$, then, by using (2.14) and elementary product manipulations, we see that (3.1) reduces to (1.5) in the notation of (2.10) as follows:

$$2\widetilde{\phi}(-q)-\widetilde{f}(q)=(-q)_\infty^{-1}\varphi(-q).$$

Entry 3.2 (Ramanujan [9, p. 2, no. 4]). *If a and b are real with* $a^2+ab+b^2=3$, *then*

$$(a+1)f_{-a}(q)+(b+1)f_{-b}(q)-(a+b-1)f_{a+b}(q)$$
$$=3\frac{(q^3;q^3)_\infty^2}{(q;q)_\infty}\prod_{n=1}^{\infty}\frac{1}{1+ab(a+b)q^{3n}+q^{6n}}. \tag{3.2}$$

In (3.2), take $a=b=1$ and use (2.14); then one obtains (1.6) in the notation of (2.10) as

$$4\widetilde{\chi}(q)-\widetilde{f}(q)=3(q)_\infty^{-1}\varphi^2(-q^3).$$

We changed the notation that Ramanujan used in the left-hand side of the next entry to avoid confusion. Also note that the series on the right side below is $f_{\sqrt{3}}(q)$ in the notation of (2.21).

Entry 3.3 (Ramanujan [9, p. 17, no. 5]). *With* $f_a(q)$ *defined by* (2.21),

$$\frac{1+\sqrt{3}}{2}f_{-1}(-q)+\frac{3+\sqrt{3}}{6}f_1(-q)$$
$$=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(1+\sqrt{3}q+q^2)\cdots(1+\sqrt{3}q^n+q^{2n})}$$
$$+\frac{2}{\sqrt{3}}\psi(-q)\frac{(q^4;q^4)_\infty}{(q^6;q^6)_\infty}\prod_{n=1}^{\infty}\frac{1}{1+\sqrt{3}q^n+q^{2n}}. \tag{3.3}$$

Entry 3.4 (Ramanujan [9, p. 17, no. 6]). *With* $\widetilde{\phi}(q)$ *defined by* (1.2),

$$\frac{1}{2}(1+e^{\pi i/4})\widetilde{\phi}(iq)+\frac{1}{2}(1+e^{-\pi i/4})\widetilde{\phi}(-iq)$$
$$=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(1+\sqrt{2}q+q^2)\cdots(1+\sqrt{2}q^n+q^{2n})}$$
$$+\frac{1}{\sqrt{2}}\psi(-q)(-q^2;q^4)_\infty\prod_{n=1}^{\infty}\frac{1}{1+\sqrt{2}q^n+q^{2n}}. \tag{3.4}$$

Note that the series on the right side above is $f_{\sqrt{2}}(q)$ in the notation of (2.21).

4. Proof of Entry 3.1

Let $a=2\cos(\theta)$, $b=2\sin(\theta)$, and $t=e^{i\theta}$. Then, it is easy to verify that

$$f_a(q)=G(-t,q),\quad f_{-a}(q)=G(t,q),\quad f_b(q)=G(it,q) \tag{4.1}$$

and

$$\frac{b-a+2}{4}=-\frac{1-i}{4t}(1-it)(1-t),\ \frac{b+a+2}{4}=\frac{1+i}{4t}(1-it)(1+t),$$
$$\frac{b}{2}=\frac{i}{2t}(1-t^2),\ a^2b^2-2=-2\cos(4\theta)=-(t^4+t^{-4}). \tag{4.2}$$

Using (4.1) and (4.2), we can rewrite (3.1) as

$$\frac{(i-1)}{4t}(1-it)(1-t)G(-t,-q)+\frac{(1+i)}{4t}(1+t)(1-it)G(t,-q)$$
$$-\frac{i}{2t}(1-t^2)G(it,q)$$
$$=\frac{(q^4;q^4)_\infty(-itq)_\infty(it^{-1}q)_\infty}{(-q;q^2)_\infty(t^4q^4;q^4)_\infty(t^{-4}q^4;q^4)_\infty}. \tag{4.3}$$

Multiplying both sides of (4.3) by $1+it$, we obtain

$$\frac{(i-1)(1-t^4)}{4t}\left(\frac{1}{1+t}G(-t,-q)-\frac{i}{1-t}G(t,-q)+\frac{i-1}{1-it}G(it,q)\right)$$
$$=\frac{(1+it)(q^4;q^4)_\infty(-itq)_\infty(it^{-1}q)_\infty}{(-q;q^2)_\infty(t^4q^4;q^4)_\infty(t^{-4}q^4;q^4)_\infty}. \tag{4.4}$$

Using definition (2.32) and dividing both sides of (4.4) by $(i-1)(1-t^4)/(4t)$, we see that (3.1) is equivalent to the identity

$$\begin{aligned}
&V(-t,-q) - iV(t,-q) + (i-1)V(it,q) \\
&\quad = -2(1+i)t\frac{(q^4;q^4)_\infty(1+it)(-itq)_\infty(it^{-1}q)_\infty}{(-q;q^2)_\infty(1-t^4)(t^4q^4;q^4)_\infty(t^{-4}q^4;q^4)_\infty} \\
&\quad = -2(1+i)t\frac{(q^4;q^4)_\infty(-it)_\infty(it^{-1}q)_\infty}{(-q;q^2)_\infty(t^4;q^4)_\infty(t^{-4}q^4;q^4)_\infty} \\
&\quad = -2(1+i)t\frac{(q^4;q^4)^2_\infty f(it,-it^{-1}q)}{(-q;q^2)_\infty(q;q)_\infty f(-t^4,-t^{-4}q^4)},
\end{aligned} \tag{4.5}$$

where in the last step we used the Jacobi triple product identity (2.9). We will verify that (4.5) is valid for $|q|<|t|<|q^{-1}|$ for any fixed $|q|<1$. Let

$$L(t,q) := V(-t,-q) - iV(t,-q) + (i-1)V(it,q),$$

$$R(t,q) := -2(1+i)t\frac{(q^4;q^4)^2_\infty f(it,-it^{-1}q)}{(-q;q^2)_\infty(q;q)_\infty f(-t^4,-t^{-4}q^4)}.$$

The proof of Entry 3.1 will be complete once we show that $R(t,q)-L(t,q)\equiv 0$. This will be achieved by showing that $R(t,q)-L(t,q)$ satisfies a q-difference equation of the sort stated in Lemma 2.3 and has no poles, thereby, forcing it to vanish identically.

Note that if we define $k(z) := f(cz, c^{-1}z^{-1}q)$, then by (2.6) we have

$$\frac{k(zq)}{k(z)} = \frac{f(czq,c^{-1}z^{-1})}{f(cz,c^{-1}z^{-1}q)} = \frac{c^{-1}z^{-1}f(cz,c^{-1}z^{-1}q)}{f(cz,c^{-1}z^{-1}q)} = (cz)^{-1}. \tag{4.6}$$

Following the same reasoning of (4.6), we obtain

$$\frac{R(tq,q)}{R(t,q)} = \frac{tq}{t}\frac{\dfrac{f(itq,-it^{-1})}{f(it,-it^{-1}q)}}{\dfrac{f(-t^4q^4,-t^{-4})}{f(-t^4,-t^{-4}q^4)}} = q\frac{(it)^{-1}}{(-t^4)^{-1}} = iqt^3.$$

Let us verify now that $L(t,q)$ also satisfies the same q-difference equation. To that end,

$$\begin{aligned}
&L(tq,q) - iqt^3L(t,q) \\
&\quad = V(-tq,-q) - iV(tq,-q) + (i-1)V(itq,q) \\
&\qquad - iqt^3\{V(-t,-q) - iV(t,-q) + (i-1)V(it,q)\}
\end{aligned}$$

$$
\begin{aligned}
&= \{V(-tq,-q) - qt^3 V(t,-q)\} - i\{V(tq,-q) + qt^3 V(-t,-q)\} \\
&\quad + (i-1)\{V(itq,q) - iqt^3 V(it,q)\} \\
&= 1 - (-q)t^2 - i(1-(-q)(-t)^2) + (i-1)(1-q(it)^2) \\
&= 1 + qt^2 - i(1+qt^2) + (i-1)(1+qt^2) = 0,
\end{aligned}
$$

where we employed (2.33). Now Lemma 2.3 implies that $R(t,q) - V(t,q)$ either has at least 3 poles in the region $|q| < |z| \leqslant 1$, or vanishes identically. But $R(t,q) - V(t,q)$ has at most 3 poles, namely at $t = 1, -1$, and $-i$ in that region, and they are all removable as we shall demonstrate. It suffices to show that $t = 1$ is a removable singularity. Thus,

$$
\begin{aligned}
\lim_{t\to 1}(1-t)L(t) &= \lim_{t\to 1}(1-t)\,\{V(-t,-q) - iV(t,-q) + (i-1)V(it,q)\} \\
&= \lim_{t\to 1}(1-t)\left\{\frac{1}{1+t}G(-t,-q) - \frac{i}{1-t}G(t,-q) + \frac{i-1}{1-it}G(it,q)\right\} \\
&= -i\lim_{t\to 1} G(t,-q) = -i(-q;-q)_\infty^{-1},
\end{aligned}
\tag{4.7}
$$

by (2.23).

Next, by two applications of (2.9) and (2.20),

$$
\begin{aligned}
&\lim_{t\to 1}(1-t)R(t) \\
&= \lim_{t\to 1}\,(1-t)\left\{-2(1+i)t\frac{(q^4;q^4)_\infty^2 f(it,-it^{-1}q)}{(-q;q^2)_\infty(q;q)_\infty f(-t^4,-t^{-4}q^4)}\right\} \\
&= -2(1+i)\lim_{t\to 1}\,(1-t)t\frac{(q^4;q^4)_\infty^2 f(it,-it^{-1}q)}{(-q;q^2)_\infty(q;q)_\infty(t^4;q^4)_\infty(t^{-4}q^4;q^4)_\infty(q^4;q^4)_\infty} \\
&= -2(1+i)\lim_{t\to 1}\,(1-t)t\frac{(q^4;q^4)_\infty f(it,-it^{-1}q)}{(-q;q^2)_\infty(q;q)_\infty(1-t^4)(t^4q^4;q^4)_\infty(t^{-4}q^4;q^4)_\infty} \\
&= -\frac{(1+i)(q^4;q^4)_\infty f(i,-iq)}{2(-q;q^2)_\infty(q;q)_\infty(q^4;q^4)_\infty(q^4;q^4)_\infty} = -\frac{(1+i)f(i,-iq)}{2(-q;q^2)_\infty(q;q)_\infty(q^4;q^4)_\infty} \\
&= -\frac{(1+i)(-i;q)_\infty(iq;q)_\infty(q;q)_\infty}{2(-q;q^2)_\infty(q;q)_\infty(q^4;q^4)_\infty} = -\frac{(1+i)(1+i)(-iq;q)_\infty(iq;q)_\infty}{2(-q;q^2)_\infty(q^4;q^4)_\infty} \\
&= -i\frac{(-q^2;q^2)_\infty}{(-q;q^2)_\infty(q^4;q^4)_\infty} = -\frac{i}{(-q;q^2)_\infty(q^4;q^4)_\infty(q^2;q^4)_\infty} \\
&= -\frac{i}{(-q;q^4)_\infty(-q^3;q^4)_\infty(q^4;q^4)_\infty(q^2;q^4)_\infty} = -\frac{i}{(-q;-q)_\infty}.
\end{aligned}
\tag{4.8}
$$

Hence, by (4.7) and (4.8), $L(t,q) - R(t,q)$ has a removable singularity at $t = 1$. By our earlier remarks, this completes the proof of Entry 3.1.

5. Proof of Entry 3.2

Our proof of Entry 3.2 is similar to our proof of Entry 3.1. Since $3 = a^2 + ab + b^2 = (a-b)^2 + 3ab = (a+b)^2 - ab$, we must have $|ab| < 4$. Assume without lost of generality that $|a| < |b|$, and let $a = 2\cos(\theta)$. Solving $a^2 + ab + b^2 = 3$ for b gives $b = -\cos(\theta) \mp \sqrt{3}\sin(\theta)$. We will take $b = -\cos(\theta) + \sqrt{3}\sin(\theta) = 2\sin(\theta - \pi/6)$, since replacing θ by $-\theta$ gives the other value for b.

Let $t = e^{i\theta}$ and $\rho = e^{2\pi i/3}$. Using this parametrization we obtain

$$a = t + t^{-1}, \quad b = \rho^{-1}t + \rho t^{-1}, \quad \text{and} \quad a + b = -\rho t - \rho^{-1}t^{-1},$$

which, in turn, implies that

$$f_{-a}(q) = G(t), \quad f_{-b}(q) = G(\rho^{-1}t), \quad \text{and} \quad f_{a+b}(q) = G(\rho t).$$

One can easily verify that

$$a + 1 = \frac{1-t^3}{t(1-t)}, \quad b + 1 = \frac{\rho(1-t^3)}{t(1-\rho^{-1}t)}, \quad \text{and} \quad a + b - 1 = -\frac{\rho^{-1}(1-t^3)}{t(1-\rho t)}.$$

Now, the left side of (3.2) which we recall below, becomes

$$\begin{aligned}
&(a+1)f_{-a}(q) + (b+1)f_{-b}(q) - (a+b-1)f_{a+b}(q) \\
&= \frac{1-t^3}{t(1-t)}G(t) + \frac{\rho(1-t^3)}{t(1-\rho^{-1}t)}G(\rho^{-1}t) + \frac{\rho^{-1}(1-t^3)}{t(1-\rho t)}G(\rho t) \\
&= \frac{1-t^3}{t}(V(t) + \rho V(\rho^{-1}t) + \rho^{-1}V(\rho t)).
\end{aligned}$$

While the right-hand side of (3.2), after observing that

$$ab(a+b) = -2\cos(3\theta) = -(t^3 + t^{-3}),$$

reduces to

$$\frac{3(q^3;q^3)_\infty^2}{(q;q)_\infty (t^3q^3;q^3)_\infty (t^{-3}q^3;q^3)_\infty}.$$

Thus, Entry 3.2 is equivalent, by (2.9), to the identity

$$V(t) + \rho V(\rho^{-1}t) + \rho^{-1}V(\rho t) = \frac{3t(q^3;q^3)_\infty^3}{f(-q)f(-t^3, -t^{-3}q^3)}. \tag{5.1}$$

Let $N(t)$ and $D(t)$ denote the right and left sides of (5.1), respectively. We will verify that $N(t)-D(t)$ satisfies the q-difference equation $N(tq)-D(tq)=-qt^3(N(t)-D(t))$ without any poles in $|q|<|t|\leqslant 1$. Then using Lemma 2.3, we conclude that $N(t)-D(t)\equiv 0$.

We employ (4.6) with $c=-1$, and t and q replaced by t^3 and q^3, respectively, to deduce that

$$\frac{N(tq)}{N(t)}=\frac{tq}{t}\frac{1}{(-t^3)^{-1}}=-qt^3.$$

Next,

$$\begin{aligned}
&D(tq)+qt^3D(t)\\
&\quad=V(tq)+\rho V(\rho^{-1}tq)+\rho^{-1}V(\rho tq)+qt^3\{V(t)+\rho V(\rho^{-1}t)+\rho^{-1}V(\rho t)\}\\
&\quad=V(tq)+qt^3V(t)+\rho\{V(\rho^{-1}tq)+qt^3V(\rho^{-1}t)\}+\rho^{-1}\{V(\rho tq)+qt^3V(\rho t)\}\\
&\quad=1-qt^2+\rho(1-q(\rho^{-1}t)^2)+\rho^{-1}(1-q(\rho t)^2)\\
&\quad=1+\rho+\rho^{-1}-qt^2(1+\rho^{-1}+\rho)=0,
\end{aligned}$$

where we used (2.33). Lemma 2.3 now implies that either $N(t)-D(t)$ vanishes or has three more poles than zeros in $|q|<|t|\leqslant 1$. But $N(t)-D(t)$ has at most three poles, namely at $t=1,\rho,\rho^{-1}$, and they are all removable as we demonstrate. It suffices to show that $t=1$ is removable.

By (2.23),

$$\begin{aligned}
\lim_{t\to 1}(1-t)D(t)&=\lim_{t\to 1}(1-t)\{V(t)+\rho V(\rho^{-1}t)+\rho^{-1}V(\rho t)\}\\
&=\lim_{t\to 1}(1-t)\left\{\frac{1}{1-t}G(t)+\rho\frac{1}{1-\rho^{-1}t}G(\rho^{-1}t)+\rho^{-1}\frac{1}{1-\rho t}G(\rho t)\right\}\\
&=\lim_{t\to 1}G(t)=\frac{1}{f(-q)}.
\end{aligned}$$

By the Jacobi triple product identity (2.9),

$$\begin{aligned}
\lim_{t\to 1}\,(1-t)N(t)&=\lim_{t\to 1}\,(1-t)\frac{3t(q^3;q^3)_\infty^3}{f(-q)f(-t^3,-t^{-3}q^3)}\\
&=\lim_{t\to 1}\,(1-t)\frac{3t(q^3;q^3)_\infty^2}{f(-q)(1-t^3)(t^3q^3;q^3)_\infty(t^{-3}q^3;q^3)_\infty}=\frac{1}{f(-q)}.
\end{aligned}$$

We have shown that $N(t)-D(t)$ has a removable singularity at $t=1$. By our earlier remarks this completes the proof of Entry 3.2.

6. Proof of Entry 3.3

If $a=1$, $b=\sqrt{3}$ in Entry 3.1, then

$$\begin{aligned}&\frac{\sqrt{3}-1+2}{4}f_1(-q)+\frac{\sqrt{3}+1+2}{4}f_{-1}(-q)-\frac{\sqrt{3}}{2}f_{\sqrt{3}}(q)\\&\quad=\frac{(q^4;q^4)_\infty}{(-q;q^2)_\infty}\prod_{n=1}^{\infty}\frac{1-\sqrt{3}q^n+q^{2n}}{1+q^{4n}+q^{8n}}.\end{aligned}$$

Multiplying both sides by $2/\sqrt{3}$, we find that

$$\begin{aligned}&\frac{3+\sqrt{3}}{6}f_1(-q)+\frac{1+\sqrt{3}}{2}f_{-1}(-q)\\&\quad=f_{\sqrt{3}}(q)+\frac{2}{\sqrt{3}}\frac{(q^4;q^4)_\infty}{(-q;q^2)_\infty}\prod_{n=1}^{\infty}\frac{1-\sqrt{3}q^n+q^{2n}}{1+q^{4n}+q^{8n}}.\end{aligned}$$

We need to show then that

$$\frac{2}{\sqrt{3}}\psi(-q)\frac{(q^4;q^4)_\infty}{(q^6;q^6)_\infty}\prod_{n=1}^{\infty}\frac{1}{1+\sqrt{3}q^n+q^{2n}}=\frac{2}{\sqrt{3}}\frac{(q^4;q^4)_\infty}{(-q;q^2)_\infty}\prod_{n=1}^{\infty}\frac{1-\sqrt{3}q^n+q^{2n}}{1+q^{4n}+q^{8n}}. \tag{6.1}$$

Recall that ψ is defined by (2.11). Now,

$$\begin{aligned}&\frac{(q^6;q^6)_\infty}{(-q;q^2)_\infty}\prod_{n=1}^{\infty}\frac{(1-\sqrt{3}q^n+q^{2n})(1+\sqrt{3}q^n+q^{2n})}{1+q^{4n}+q^{8n}}\\&\quad=\frac{(q^6;q^6)_\infty}{(-q;q^2)_\infty}\prod_{n=1}^{\infty}\frac{1-q^{2n}+q^{4n}}{1+q^{4n}+q^{8n}}\\&\quad=\frac{(q^6;q^6)_\infty}{(-q;q^2)_\infty}\prod_{n=1}^{\infty}\frac{1}{1+q^{2n}+q^{4n}}\\&\quad=\frac{(q^6;q^6)_\infty(q^2;q^2)_\infty}{(-q;q^2)_\infty(q^6;q^6)_\infty}=\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}=\psi(-q),\end{aligned}$$

where in the last step (2.13) is used. Equality (6.1) now follows, and so the proof of Entry 3.3 is complete.

7. Proof of Entry 3.4

Let $\alpha = e^{i\pi/4}$. Clearly, using the notation of (2.22), we have $\widetilde{\phi}(q) = G(i, q)$, and $f_{\sqrt{2}}(q) = G(-\alpha, q)$. We can then restate Entry 3.4 as

$$\frac{1+\alpha}{2}G(i, iq) + \frac{1+\alpha^{-1}}{2}G(i, -iq) - G(-\alpha, q) = \frac{1}{\sqrt{2}}\frac{\psi(-q)(-q^2; q^4)_\infty}{(-\alpha q)_\infty(-\alpha^{-1}q)_\infty}.$$

Dividing both sides by $(1+\alpha)/2$ and employing (2.9), we arrive at

$$G(i, iq) + \alpha^{-1}G(i, -iq) - \frac{2}{1+\alpha}G(-\alpha, q) = \sqrt{2}\frac{\psi(-q)f(-q)(-q^2; q^4)_\infty}{f(\alpha, \alpha^{-1}q)}. \tag{7.1}$$

If we replace q by iq, (7.1) becomes

$$G(i, -q) + \alpha^{-1}G(i, q) - 2V(-\alpha, iq) = \sqrt{2}\frac{\psi(-iq)f(-iq)(q^2; q^4)_\infty}{f(\alpha, \alpha q)}. \tag{7.2}$$

The following identities will be needed for the remainder of the proof:

$$f(\alpha, \alpha^{-1}q)f(-\alpha, -\alpha^{-1}q) = (1-i)(-q^4; q^4)_\infty f^2(-q), \tag{7.3}$$

$$f(\alpha, \alpha q)f(-\alpha, -\alpha q) = (1-i)(-q^4; q^4)_\infty f^2(-iq), \tag{7.4}$$

$$f(\alpha, \alpha^{-1}q) = \psi(iq) + \alpha\psi(-iq), \tag{7.5}$$

$$f(-\alpha, -\alpha^{-1}q) = \psi(iq) - \alpha\psi(-iq), \tag{7.6}$$

$$f(\alpha, \alpha q) = \psi(-q) + \alpha\psi(q), \tag{7.7}$$

$$f(-\alpha, -\alpha q) = \psi(-q) - \alpha\psi(q). \tag{7.8}$$

We now offer proofs for all six identities.

To prove (7.3) we employ (2.9) to find that

$$\begin{aligned}
&f(\alpha,\alpha^{-1}q)f(-\alpha,-\alpha^{-1}q) = (-\alpha)_\infty(-\alpha^{-1}q)_\infty f(-q)(\alpha)_\infty(\alpha^{-1}q)_\infty f(-q)\\
&\quad = (i;q^2)_\infty(-iq^2;q^2)_\infty f^2(-q)\\
&\quad = (1-i)(-q^4;q^4)_\infty f^2(-q).
\end{aligned}$$

Clearly, (7.4) is obtained by replacing q by iq in (7.3). Recall that $\psi(q)=f(q,q^3)$. From (2.7) and (2.8),

$$\begin{aligned}
&f(\alpha,\alpha^{-1}q)+f(-\alpha,-\alpha^{-1}q)\\
&\quad = 2f(\alpha^2q,\alpha^{-2}q^3) = 2f(iq,-iq^3) = 2\psi(iq),
\end{aligned}\tag{7.9}$$

$$\begin{aligned}
&f(\alpha,\alpha^{-1}q)-f(-\alpha,-\alpha^{-1}q)\\
&\quad = 2\alpha f(\alpha^{-2}q,\alpha^2q^3) = 2\alpha f(-iq,iq^3) = 2\alpha\psi(-iq)
\end{aligned}\tag{7.10}$$

Equalities (7.9) and (7.10) readily imply (7.5) and (7.6). And finally we obtain (7.7) and (7.8) by replacing q by iq in (7.5) and (7.6), respectively.

We now return to (7.2) and use (7.4), (7.8), and (2.13) with q replaced by iq to deduce that

$$\begin{aligned}
&G(i,-q)+\alpha^{-1}G(i,q)-2V(-\alpha,iq)\\
&\quad = \sqrt{2}\frac{\psi(-iq)f(-iq)(q^2;q^4)_\infty}{f(\alpha,\alpha q)}\\
&\quad = \sqrt{2}\frac{\psi(-iq)f(-iq)(q^2;q^4)_\infty f(-\alpha,-\alpha q)}{f(\alpha,\alpha q)f(-\alpha,-\alpha q)}\\
&\quad = \sqrt{2}\frac{\psi(-iq)f(-iq)(q^2;q^4)_\infty(\psi(-q)-\alpha\psi(q))}{(1-i)(-q^4;q^4)_\infty f^2(-iq)}\\
&\quad = \alpha\frac{\psi(-iq)(q^2;q^4)_\infty(\psi(-q)-\alpha\psi(q))}{(-q^4;q^4)_\infty f(-iq)}\\
&\quad = \alpha\frac{(q^2;q^4)_\infty(\psi(-q)-\alpha\psi(q))}{(-q^4;q^4)_\infty(-q^2;q^4)_\infty}\\
&\quad = \alpha\frac{(q^2;q^4)_\infty(\psi(-q)-\alpha\psi(q))}{(-q^2;q^2)_\infty}\\
&\quad = \alpha(q^2;q^4)^2_\infty(\psi(-q)-\alpha\psi(q)).
\end{aligned}\tag{7.11}$$

It suffices now to prove (7.11).

Let

$$K(t,q) := \alpha V(it,iq) - \alpha V(-it,iq) + iV(t,iq) + iV(-t,iq) \\ + (1-i)V(-\alpha t,-q) - (1+i)V(\alpha t,-q). \tag{7.12}$$

The identity,

$$K(t,q) = -4\alpha^{-1}t\frac{f^3(-q^4)f(\alpha^{-1}t,\alpha t^{-1}q)}{f(-iq)f(\alpha,\alpha^{-1}q)f(-t^4,-t^{-4}q^4)} \\ -2(1+i)t\frac{\psi(-q^2)f^2(-q)f(\alpha^{-1}t,\alpha t^{-1}q)}{f(t,t^{-1}q)f(-it,it^{-1}q)f(-it^2,it^{-2}q^2)}, \tag{7.13}$$

together with Entry 3.1 will be used to verify (7.11). We will not prove (7.13), because its proof is very similar to that of (5.1). The q-difference equation satisfied by $K(t,q)$ is $K(tq,q) = -\alpha qt^3K(t,q)$. It then suffices, by Lemma 2.3, to verify that the residues at four of the six singularities match those of the two representations (7.12) and (7.13) of $K(t,q)$. It is easily verified that $t = -\alpha$ is a zero for the two representations (7.12) and (7.13) of $K(t,q)$. Therefore, one only needs to check the residues at any three of the six singularities. If we knew the two other zeros whose existence is guaranteed by Lemma 2.3, we then would be able to reduce the right-hand side of (7.13) to a single product, but we are unable to determine these two zeros.

Let us define, by using (4.5),

$$E(t,q) \coloneqq V(-t,-q) - iV(t,-q) + (i-1)V(it,q) \tag{7.14}$$

$$= -2(1+i)t\frac{(q^4;q^4)^2_\infty f(it,-it^{-1}q)}{(-q;q^2)_\infty(q;q)_\infty f(-t^4,-t^{-4}q^4)}. \tag{7.15}$$

We will verify by using (7.12) and (7.14) that

$$G(i,-q) + \alpha^{-1}G(i,q) - 2V(-\alpha,iq) \\ = \frac{1}{\alpha(1-i)}(E(\alpha,iq) + E(-\alpha,iq)) + \frac{1}{2}iK(\alpha,q) + \frac{1}{2}\alpha K(\alpha,-q). \tag{7.16}$$

Equalities (7.13) and (7.15) will then be used to verify that (7.16) reduces to the right-hand side of (7.11), which will complete the proof of Entry 3.4.

Using (7.14), we have

$$E(t,q) + E(-t,q) = (1-i)\{V(t,-q) + V(-t,-q) - V(it,q) - V(-it,q)\}.$$

Setting $t = \alpha$, we find that

$$\begin{aligned}
&E(\alpha, q) + E(-\alpha, q) \\
&\quad = (1-i)\{V(\alpha, -q) + V(-\alpha, -q) - V(i\alpha, q) - V(-i\alpha, q)\} \\
&\quad = (1-i)\{V(\alpha, -q) + V(-\alpha, -q) - V(-\alpha^{-1}, q) - V(\alpha^{-1}, q)\} \\
&\quad = (1-i)\{V(\alpha, -q) + V(-\alpha, -q) - \alpha V(-\alpha, q) + \alpha V(\alpha, q)\}.
\end{aligned}$$

Replacing q by iq and dividing by $\alpha(1-i)$, we obtain

$$\begin{aligned}
&\frac{1}{\alpha(1-i)}(E(\alpha, iq) + E(-\alpha, iq)) \\
&\quad = \alpha^{-1} V(\alpha, -iq) + \alpha^{-1} V(-\alpha, -iq) - V(-\alpha, iq) + V(\alpha, iq).
\end{aligned} \tag{7.17}$$

By (7.12),

$$\begin{aligned}
K(\alpha, q) &= \alpha V(i\alpha, iq) - \alpha V(-i\alpha, iq) + iV(\alpha, iq) + iV(-\alpha, iq) \\
&\quad + (1-i)V(-i, -q) - (1+i)V(i, -q) \\
&= \alpha V(-\alpha^{-1}, iq) - \alpha V(\alpha^{-1}, iq) + iV(\alpha, iq) + iV(-\alpha, iq) \\
&\quad + (1-i)V(-i, -q) - (1+i)V(i, -q) \\
&= iV(-\alpha, iq) + iV(\alpha, iq) + iV(\alpha, iq) + iV(-\alpha, iq) \\
&\quad - i(1-i)V(i, -q) - (1+i)V(i, -q) \\
&= 2iV(\alpha, iq) + 2iV(-\alpha, iq) - 2iG(i, -q).
\end{aligned} \tag{7.18}$$

Combining (7.17) and (7.18), we find that

$$\begin{aligned}
&\frac{1}{\alpha(1-i)}(E(\alpha, iq) + E(-\alpha, iq)) + \frac{1}{2} iK(\alpha, q) + \frac{1}{2}\alpha K(\alpha, -q) \\
&\quad = \alpha^{-1} V(\alpha, -iq) + \alpha^{-1} V(-\alpha, -iq) - V(-\alpha, iq) + V(\alpha, iq) \\
&\qquad - V(\alpha, iq) - V(-\alpha, iq) + G(i, -q) \\
&\qquad - \alpha^{-1} V(\alpha, -iq) - \alpha^{-1} V(-\alpha, -iq) + \alpha^{-1} G(i, q) \\
&\quad = G(i, -q) + \alpha^{-1} G(i, q) - 2V(-\alpha, iq).
\end{aligned}$$

This proves our first claim that (7.16) holds.

Using (7.13), (2.3), (2.6), and (2.19) with q replaced by q^4, we find that

$$\begin{aligned}
K(\alpha, q) = & -4\frac{f^3(-q^4)f(1,q)}{f(-iq)f(\alpha,\alpha^{-1}q)f(1,q^4)} \\
& -2\alpha(1+i)\frac{\psi(-q^2)f^2(-q)f(1,q)}{f(\alpha,\alpha^{-1}q)f(-i\alpha, i\alpha^{-1}q)f(1,q^2)} \\
= & -4\frac{f^3(-q^4)\psi(q)}{f(-iq)f(\alpha,\alpha^{-1}q)\psi(q^4)} - 2\alpha(1+i)\frac{\psi(-q^2)f^2(-q)\psi(q)}{f(\alpha,\alpha^{-1}q)f(\alpha^{-1},\alpha q)\psi(q^2)} \\
= & -4\frac{f^3(-q^4)\psi(q)}{f(-iq)f(\alpha,\alpha^{-1}q)\psi(q^4)} + 2(1-i)\frac{\psi(-q^2)f^2(-q)\psi(q)}{f^2(\alpha,\alpha^{-1}q)\psi(q^2)} \\
= & -4\frac{\varphi^2(-q^4)\psi(q)}{f(-iq)f(\alpha,\alpha^{-1}q)} + 2(1-i)\frac{\psi(-q^2)f^2(-q)\psi(q)f(-\alpha,-\alpha^{-1}q)}{f^2(\alpha,\alpha^{-1}q)\psi(q^2)f(-\alpha,-\alpha^{-1}q)}.
\end{aligned}$$

Using (7.3) and (7.6) above, we deduce that

$$\begin{aligned}
K(\alpha, q) = & -4\frac{\varphi^2(-q^4)\psi(q)}{f(-iq)f(\alpha,\alpha^{-1}q)} \\
& +2(1-i)\frac{\psi(-q^2)f^2(-q)\psi(q)(\psi(iq)-\alpha\psi(-iq))}{f(\alpha,\alpha^{-1}q)\psi(q^2)(1-i)(-q^4;q^4)_\infty f^2(-q)} \\
= & -4\frac{\varphi^2(-q^4)\psi(q)}{f(-iq)f(\alpha,\alpha^{-1}q)} + 2\frac{\psi(-q^2)\psi(q)\psi(iq)}{f(\alpha,\alpha^{-1}q)\psi(q^2)(-q^4;q^4)_\infty} \\
& -2\alpha\frac{\psi(-q^2)\psi(q)\psi(-iq)}{f(\alpha,\alpha^{-1}q)\psi(q^2)(-q^4;q^4)_\infty} \\
= & -4\frac{\varphi^2(-q^4)\psi(q)}{f(-iq)f(\alpha,\alpha^{-1}q)} + 2\frac{\psi(-q^2)\psi(q)f^2(q^2)}{f(-iq)f(\alpha,\alpha^{-1}q)\psi(q^2)(-q^4;q^4)_\infty} \\
& -2\alpha\frac{\psi(-q^2)\psi(q)f^2(q^2)}{f(iq)f(\alpha,\alpha^{-1}q)\psi(q^2)(-q^4;q^4)_\infty},
\end{aligned} \tag{7.19}$$

where we used (2.17) in the form $f(q)\psi(-q) = f^2(-q^2)$ with q replaced by iq and $-iq$, respectively. But by (2.16),

$$\begin{aligned}
\frac{\psi(-q^2)f^2(q^2)}{\psi(q^2)(-q^4;q^4)_\infty} &= \frac{f(q^2)\psi(q^2)f(-q^2)}{\psi(q^2)(-q^4;q^4)_\infty} = \frac{(-q^2;-q^2)_\infty(q^2;q^2)_\infty}{(-q^4;q^4)_\infty} \\
&= \frac{(-q^2;q^4)_\infty(q^4;q^4)_\infty(q^2;q^4)_\infty(q^4;q^4)_\infty}{(-q^4;q^4)_\infty} = \frac{(q^4;q^8)_\infty(q^4;q^4)^2_\infty}{(-q^4;q^4)_\infty} \\
&= \frac{(q^4;q^4)^2_\infty}{(-q^4;q^4)^2_\infty} = \varphi^2(-q^4),
\end{aligned} \tag{7.20}$$

where we used Euler's identity (2.20), and (2.14). Using (7.20) in (7.19), (2.17) in the form $f(q)\psi(-q) = f^2(-q^2)$ with q replaced by iq and $-iq$, respectively, (7.5), and

(2.14), we deduce that

$$\begin{aligned}
K(\alpha,q) =& -4\frac{\varphi^2(-q^4)\psi(q)}{f(-iq)f(\alpha,\alpha^{-1}q)}+2\frac{\varphi^2(-q^4)\psi(q)}{f(-iq)f(\alpha,\alpha^{-1}q)}-2\alpha\frac{\varphi^2(-q^4)\psi(q)}{f(iq)f(\alpha,\alpha^{-1}q)}\\
=& -2\frac{\varphi^2(-q^4)\psi(q)}{f(\alpha,\alpha^{-1}q)}\left(\frac{1}{f(-iq)}+\alpha\frac{1}{f(iq)}\right)\\
=& -2\frac{\varphi^2(-q^4)\psi(q)}{f(\alpha,\alpha^{-1}q)f^2(q^2)}(\psi(iq)+\alpha\psi(-iq))\\
=& -2\frac{\varphi^2(-q^4)\psi(q)}{f^2(q^2)}=-2\frac{(q^4;q^4)_\infty^2\psi(q)}{(-q^4;q^4)_\infty^2(-q^2;-q^2)_\infty^2}\\
=& -2\frac{(q^4;q^4)_\infty^2\psi(q)}{(-q^4;q^4)_\infty^2(-q^2;q^4)_\infty^2(q^4;q^4)_\infty^2}=-2\frac{\psi(q)}{(-q^2;q^2)_\infty^2}\\
=& -2(q^2;q^4)_\infty^2\psi(q), \qquad (7.21)
\end{aligned}$$

where in the last step we used (2.20). Thus, by (7.21),

$$\frac{1}{2}iK(\alpha,q)+\frac{1}{2}\alpha K(\alpha,-q)=-i(q^2;q^4)_\infty^2\psi(q)-\alpha(q^2;q^4)_\infty^2\psi(-q). \qquad (7.22)$$

Let us evaluate now $E(\alpha,q)$. By (2.3), (2.6), and (7.15),

$$\begin{aligned}
E(\alpha,q) =& -2(1+i)\alpha\frac{(q^4;q^4)_\infty^2 f(i\alpha,i\alpha^{-1}q)}{(q;q)_\infty(-q;q^2)_\infty f(1,q^4)}\\
=& -2(1+i)\alpha\frac{(q^4;q^4)_\infty^2 f(-\alpha^{-1},-\alpha q)}{(q;q)_\infty(-q;q^2)_\infty f(1,q^4)}\\
=& (1+i)\frac{(q^4;q^4)_\infty^2 f(-\alpha,-\alpha^{-1}q)}{(q;q)_\infty(-q;q^2)_\infty\psi(q^4)}. \qquad (7.23)
\end{aligned}$$

Employ (2.11) and (2.20) to deduce that

$$\begin{aligned}
\frac{(q^4;q^4)_\infty^2}{(q;q)_\infty(-q;q^2)_\infty\psi(q^4)} =& \frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty(q;q^2)_\infty(-q;q^2)_\infty(-q^4;q^4)_\infty^2(q^4;q^4)_\infty}\\
=& \frac{(q^4;q^4)_\infty}{(q^2;q^2)_\infty(q^2;q^4)_\infty(-q^4;q^4)_\infty^2}=\frac{(q^2;q^2)_\infty(-q^2;q^2)_\infty^2}{(q^2;q^2)_\infty(-q^4;q^4)_\infty^2}\\
=& \frac{(-q^2;q^2)_\infty^2}{(-q^4;q^4)_\infty^2}=\frac{(-q^2;q^4)_\infty^2(-q^4;q^4)_\infty^2}{(-q^4;q^4)_\infty^2}=(-q^2;q^4)_\infty^2.
\end{aligned} \qquad (7.24)$$

Using (7.24) in (7.23), we obtain

$$E(\alpha,q)=(1+i)(-q^2;q^4)_\infty^2 f(-\alpha,-\alpha^{-1}q). \qquad (7.25)$$

Similarly, we obtain

$$E(-\alpha, q) = (1+i)(-q^2; q^4)_\infty^2 f(\alpha, \alpha^{-1}q). \tag{7.26}$$

Combining (7.25) and (7.26), we arrive at

$$\begin{aligned}&\frac{1}{\alpha(1-i)}\{E(\alpha, q) + E(-\alpha, q)\}\\ &\quad= \frac{1+i}{\alpha(1-i)}(-q^2; q^4)_\infty^2 (f(\alpha, \alpha^{-1}q) + f(-\alpha, -\alpha^{-1}q)) = 2\alpha(-q^2; q^4)_\infty^2 \psi(iq),\end{aligned} \tag{7.27}$$

by (7.9). Finally, replacing q by iq in (7.27), we deduce that

$$\frac{1}{\alpha(1-i)}\{E(\alpha, iq) + E(-\alpha, iq)\} = 2\alpha(q^2; q^4)_\infty^2 \psi(-q). \tag{7.28}$$

Adding (7.22) and (7.28) together, we find that (7.16) reduces to the right-hand side of (7.11), i.e.,

$$\begin{aligned}&\frac{1}{2}iK(\alpha, q) + \frac{1}{2}\alpha K(\alpha, -q) + \frac{1}{\alpha(1-i)}\{E(\alpha, iq) + E(-\alpha, iq)\}\\ &\quad= -i(q^2; q^4)_\infty^2 \psi(q) - \alpha(q^2; q^4)_\infty^2 \psi(-q) + 2\alpha(q^2; q^4)_\infty^2 \psi(-q)\\ &\quad= \alpha(q^2; q^4)_\infty^2 (\psi(-q) - \alpha\psi(q)).\end{aligned}$$

This completes the verification of (7.11), since we have already verified (7.16). Hence, the proof of Entry 3.4 is complete.

8. Proof of 1.8

Let us recall Eqs. (2.35) and (5.1), which is the equivalent form of Entry 3.2. Thus,

$$V(t) + \rho V(\rho^{-1}t) + \rho^{-1} V(\rho t) = \frac{3t(q^3; q^3)_\infty^3}{(q)_\infty f(-t^3, -t^{-3}q^3)}, \tag{8.1}$$

$$V(t) = 1 + \frac{t}{(q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n \frac{q^{3n(n+1)/2}}{1 - tq^n}, \tag{8.2}$$

where $\rho = e^{2\pi i/3}$. Using (8.2) in (8.1), we obtain

$$\begin{aligned}
&\frac{3t(q^3;q^3)_\infty^3}{(q)_\infty f(-t^3,-t^{-3}q^3)} \\
&\quad = 1 + \frac{t}{(q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{3n(n+1)/2}}{1-tq^n} \\
&\qquad + \rho + \frac{\rho t}{(q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{3n(n+1)/2}}{1-\rho^{-1}tq^n} \\
&\qquad + \rho^{-1} + \frac{\rho^{-1} t}{(q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{3n(n+1)/2}}{1-\rho tq^n} \\
&\quad = \frac{t}{(q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{3n(n+1)/2}\left\{\frac{1}{1-tq^n}+\frac{\rho}{1-\rho^{-1}tq^n}+\frac{\rho^{-1}}{1-\rho tq^n}\right\} \\
&\quad = \frac{3t}{(q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{3n(n+1)/2}}{1-t^3q^{3n}}.
\end{aligned}$$

Then, we have

$$\frac{(q^3;q^3)_\infty^3}{f(-t^3,-t^{-3}q^3)} = \sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{3n(n+1)/2}}{1-t^3q^{3n}}. \tag{8.3}$$

Now, (1.8) follows if one replaces q^3 by q and t^3 by t, respectively, and employs (2.9) in (8.3).

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